A Relational Approach to Stochastic Dominance

Bernard DE BAETS and Hans DE MEYER
Contents

- Stochastic dominance
- Transitivity of probabilistic relations
- Graded alternatives
- Non-graded alternatives
- Future directions
1. Introduction

Purpose of stochastic dominance:

- to define a *(partial) order relation* on a set of real-valued random variables (RV)
- endowed with the semantics of a *weak preference relation*:

  RV taking higher values are preferred
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- to define a (partial) order relation on a set of real-valued random variables (RV)

- endowed with the semantics of a weak preference relation:

  RV taking higher values are preferred

General principle:

- pairwise comparison of RV

- pointwise comparison of performance functions

- constructed from the distribution function
1. Performance functions

- **The cumulative distribution function (CDF) $F_X$:**
  \[ F_X(x) = \text{Prob}\{X \leq x\} \]

- **The area below the CDF $F_X$:**
  \[ G_X(x) = \int_{-\infty}^{x} F_X(t) \, dt \]
1. 1st and 2nd degree stochastic dominance (SD)

**Weak dominance relation:**

<table>
<thead>
<tr>
<th>$X \geq_{\text{FSD}} Y$</th>
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**Weak dominance relation:**

\[
X \succeq_{\text{FSD}} Y \iff F_X \leq F_Y \\
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\[
X \succeq_{\text{SSD}} Y \iff G_X \leq G_Y \\
\iff E[u(X)] \geq E[u(Y)] \\
\text{for any non-decreasing concave function } u
\]

**Strict dominance relation:**

\[
X \succ Y \iff X \succeq Y \text{ and } Y \not\succeq X
\]
1. Graphical illustration of FSD
1. Application areas

- Decision making under uncertainty

- Risk averse preference models in economics and finance:
  - e.g. in portfolio optimisation

- Social statistics:
  - e.g. in the comparison of welfare and poverty indicators

- Machine learning and multi-criteria decision making:
  - e.g. in ranking (= ordered sorting) algorithms
1. Discussion

SD induces a **crisp partial order relation** on a set of RV:

- **crisp**: no tolerance for small deviations, no grading
- **partial**: usually **sparse** graphs
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SD is theoretically attractive, but **computationally difficult**
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- does not take into account **dependence** between RV
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- SD is theoretically attractive, but **computationally difficult**

- SD uses **marginal distributions** only:
  - does not take into account **dependence** between RV

- SSD accumulates area from $-\infty$ onwards
  - introduces an **absolute reference point**
1. Main objective: graded variants of SD

- Pairwise construction of a “transitive" valued relation on a set of RV which:
  - avoids the pointwise comparison of performance functions
  - allows to incorporate dependence between the RV
  - avoids specific reference points
  - allows to induce a strict order relation on the set of RV
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- Key instrument: probabilistic relations and cycle-transitivity
2. Probabilistic relations and cycle-transitivity

Probabilistic relation $Q$ on $X: Q : X^2 \rightarrow [0, 1]$ such that

$$Q(a, b) + Q(b, a) = 1$$
2. Probabilistic relations and cycle-transitivity

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<th>$a_{abc}$</th>
<th>$\beta_{abc}$</th>
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<tr>
<td>Formula</td>
<td>$\min{Q(a, b), Q(b, c), Q(c, a)}$</td>
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A probabilistic relation $Q$ on $A$ is called cycle-transitive w.r.t. an upper bound function $U$ if for any $a, b, c \in A$

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc})$$

with the dual lower bound function $L$ defined by

$$L(\alpha, \beta, \gamma) = 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha)$$
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A function $U : \Delta = \{(x, y, z) \in [0, 1]^3 \mid x \leq y \leq z\} \rightarrow \mathbb{R}$ is called an upper bound function if it satisfies:

- $U(0, 0, 1) \geq 0$ and $U(0, 1, 1) \geq 1$
- for any $(\alpha, \beta, \gamma) \in \Delta$:

\[
U(\alpha, \beta, \gamma) \geq 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha)
\]
A probabilistic relation $Q$ on $A$ is called $g$-stochastic transitive if

$$(Q(a, b) \geq 1/2 \land Q(b, c) \geq 1/2) \Rightarrow g(Q(a, b), Q(b, c)) \leq Q(a, c)$$

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Moderate stochastic transitivity: equivalent to transitivity of cut relations $Q_\alpha$, $\alpha \geq 1/2$
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Moderate stochastic transitivity: equivalent to transitivity of cut relations $Q_\alpha$, $\alpha \geq 1/2$

Partial stochastic transitivity: a weaker variant of moderate stochastic transitivity
2. General framework indeed

Commutative, increasing function $g$ such that $g(1/2, x) \leq x$

Theorem: $g$-stochastic transitivity = cycle-transitivity w.r.t.

$$U_g(\alpha, \beta, \gamma) = \begin{cases} 
\beta + \gamma - g(\beta, \gamma) & , \text{if } \beta \geq 1/2 \land \alpha < 1/2 \\
1/2 & , \text{if } \alpha = 1/2 \\
2 & , \text{if } \beta < 1/2 
\end{cases}$$
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- Moderate stochastic transitivity:

$U_{ms}(\alpha, \beta, \gamma) = \begin{cases} 
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\gamma & \text{, else}
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Partial stochastic transitivity: $$U_{ps}(\alpha, \beta, \gamma) = \gamma$$
2. Fuzzy relations and $T$-transitivity

Fuzzy relation $R$ on $X$: $R : X^2 \rightarrow [0, 1]$
2. Fuzzy relations and $T$-transitivity

Fuzzy relation $R$ on $X$: $R : X^2 \to [0, 1]$

Triangular norm (t-norm): $T : [0, 1]^2 \to [0, 1]$ such that
- increasing, neutral element 1 (and absorbing element 0)
- commutative and associative
2. Fuzzy relations and $T$-transitivity

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Basic t-norms:

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- **$T$-transitivity of a fuzzy relation** $R$:
  
  $T(R(a, b), R(b, c)) \leq R(a, c)$
2. The Frank t-norm family

Prototypical solutions of the functional equation:

\[ T(x, y) + 1 - T(1 - x, 1 - y) = x + y \]
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Frank t-norm family \( (T^F_s)_{s \in [0, \infty]} \): for \( s \in ]0, 1[ \cup ]1, \infty[ \)

\[ T^F_s(x, y) = \log_s \left(1 + \frac{(s^x - 1)(s^y - 1)}{s - 1}\right) \]
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- Limit cases:

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2. General framework indeed

**1-Lipschitz t-norm** $T$:

$$|T(x_1, y_1) - T(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|$$
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Theorem: $T$-transitivity = cycle-transitivity w.r.t.

$$U_T(\alpha, \beta, \gamma) = \alpha + \beta - T(\alpha, \beta)$$

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- Frank t-norms are 1-Lipschitz
3. Pairwise comparison

Random vector \((X, Y)\): winning probabilities

\[
Q(X, Y) = \text{Prob}\{X > Y\} + \frac{1}{2} \text{Prob}\{X = Y\}
\]

leads to reciprocity: \(Q(X, Y) + Q(Y, X) = 1\)

is based on the joint distribution, and not on marginal ones
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Random vector $(X, Y)$: winning probabilities

\[ Q(X, Y) = \text{Prob}\{X > Y\} + \frac{1}{2} \text{Prob}\{X = Y\} \]

leads to reciprocity: $Q(X, Y) + Q(Y, X) = 1$

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Special cases:
- discrete random vector:
  \[ Q(X, Y) = \sum_{k > l} p_{X,Y}(k, l) + \frac{1}{2} \sum_k p_{X,Y}(k, k) \]
- continuous random vector:
  \[ Q(X, Y) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{x} f_{X,Y}(x, y) \, dy \]
3. Copulas

**Copula:** \( C : [0, 1]^2 \to [0, 1] \) such that
- neutral element 1, absorbing element 0
- moderate growth:

\[
(x_1 \leq x_2 \land y_1 \leq y_2) \Rightarrow C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1)
\]
3. Copulas

- **Copula:** $C : [0, 1]^2 \rightarrow [0, 1]$ such that
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\]

- Frank t-norms are copulas and $T_L \leq C \leq T_M$

- Relationship between t-norms and copulas:

  \[
  \text{copula + associativity } \subseteq \text{ t-norm}
  \]
  \[
  \text{t-norm + 1-Lipschitz } \subseteq \text{ copula}
  \]
3. Sklar’s theorem

Random vector \((X, Y)\): there exists a copula \(C\) s.t.

\[
F_{X,Y}(x, y) = C(F_X(x), F_Y(y))
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3. **Sklar’s theorem**

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- Captures dependence structure irrespective of the marginals
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Captures dependence structure irrespective of the marginals

Probabilistic interpretation:

| \(T_M\) | co-monotonicity |
| \(T_P\) | independence |
| \(T_L\) | counter-monotonicity |
3. Case 1: generalized dice

- \( X \) and \( Y \) uniformly distributed on multisets of cardinality \( n \)

- Order elements of multiset \( A_X \) increasingly:
  \[
  x_1 \leq x_2 \leq \cdots \leq x_n
  \]

- Order elements of multiset \( A_Y \) increasingly:
  \[
  y_1 \leq y_2 \leq \cdots \leq y_n
  \]
3. The case $C = T_P$: independent comparison

Probabilistic relation:

$$Q^P(X, Y) = \frac{1}{n^2} \sum_{k,l=1}^{n} \delta^P_{kl}$$

with

$$\delta^P_{kl} = \begin{cases} 
1 & \text{, if } x_k > y_l \\
1/2 & \text{, if } x_k = y_l \\
0 & \text{, if } x_k < y_l 
\end{cases}$$
3. Example

\[ Q^P(X, Y) = \frac{7}{16} \]
3. The case $C = T_M$: co-monotone comparison

Probabilistic relation:

$$Q^M(X, Y) = \frac{1}{n} \sum_{k=1}^{n} \delta^M_k$$

with

$$\delta^M_k = \begin{cases} 
1, & \text{if } x_k > y_k \\
1/2, & \text{if } x_k = y_k \\
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\end{cases}$$
3. The case $C = T_L$: counter-monotone comparison

Probabilistic relation:

$$Q^L(X, Y) = \frac{1}{n} \sum_{k=1}^{n} \delta^L_k$$

with

$$\delta^L_k = \begin{cases} 
1, & \text{if } x_k > y_{n-k+1} \\
1/2, & \text{if } x_k = y_{n-k+1} \\
0, & \text{if } x_k < y_{n-k+1}
\end{cases}$$
3. Example (continued)

\[ Q^M(X, Y) = \frac{3}{8} \quad Q^L(X, Y) = \frac{1}{2} \]
3. Case 2: continuous RV

The case $T_P$: independent RV

$$Q^P(X, Y) = E_X[F_Y]$$
3. Case 2: continuous RV

The case $T_P$: independent RV

$$Q^P(X, Y) = E_X[F_Y]$$

The case $T_M$: co-monotone RV

$$Q^M(X, Y) = \int_{x:F_X(x)<F_Y(x)} f_X(x) \, dx + \frac{1}{2} \int_{x:F_X(x)=F_Y(x)} f_X(x) \, dx$$

$Q^M(X, Y) = 1$ iff $F_X < F_Y$ where $f_X \neq 0$:

more restrictive than $\succ_{FSD}$
$Q^M(X, Y) = t_1 + t_3 + \frac{1}{2} t_2$
3. The compatibility problem

Random vector \((X_1, X_2, \ldots, X_n)\): there exist copulas \(C_{ij}\) s.t.

\[
F_{X_i,X_j}(x,y) = C_{ij}(F_{X_i}(x), F_{X_j}(y))
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The compatibility problem:

- not all combinations of copulas are possible
- all \(C_{ij} = C\) is possible for \(C \in \{T_M, T_P\}\)
- \(C_{12} = C_{13} = C_{23} = T_L\) is impossible
3. The compatibility problem

- Random vector \((X_1, X_2, \ldots, X_n)\): there exist copulas \(C_{ij}\) s.t.

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- The compatibility problem:
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  - all \(C_{ij} = C\) is possible for \(C \in \{T_M, T_P\}\)
  - \(C_{12} = C_{13} = C_{23} = T_L\) is impossible

- Artificial coupling:
  - winning probabilities require only bivariate coupling
  - copula = comparison strategy
  - does not reflect the real dependence
3. Coupling by the same copula: cycle-transitivity

Stable commutative copulas:

\[ C(x, y) + 1 - C(1 - x, 1 - y) = x + y \]
3. Coupling by the same copula: cycle-transitivity

- Stable commutative copulas:

\[ C(x, y) + 1 - C(1 - x, 1 - y) = x + y \]

- Theorem: for a stable commutative copula \( C \), the probabilistic relation \( Q^C \) is cycle-transitive w.r.t.

\[ U^C(\alpha, \beta, \gamma) = \gamma + C(\beta, 1 - \gamma) \]
3. Coupling by the same copula: cycle-transitivity

Frank t-norms are stable commutative copulas:

\[ U^s(\alpha, \beta, \gamma) = \beta + \gamma - T^F_s(\beta, \gamma) \]
3. Coupling by the same copula: cycle-transitivity

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<th>copula</th>
<th>upper bound f.</th>
<th>equivalent</th>
<th>known as</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_M )</td>
<td>( \min(\beta + \gamma, 1) )</td>
<td>( 1 )</td>
<td>( T_L )-transitivity</td>
</tr>
<tr>
<td>( T_P )</td>
<td>( \beta + \gamma - \beta\gamma )</td>
<td>( \gamma )</td>
<td>dice-transitivity</td>
</tr>
<tr>
<td>( T_L )</td>
<td>( \max(\beta, \gamma) )</td>
<td></td>
<td>partial stochastic transitivity</td>
</tr>
</tbody>
</table>
4. Statistical preference and cycles

Statistical preference: $X \succeq Y$ if $Q(X, Y) \geq 1/2$
4. Statistical preference and cycles

Statistical preference: \( X \succeq Y \) if \( Q(X, Y) \geq 1/2 \)

Theorem: FSD implies statistical preference
4. Statistical preference and cycles

- **Statistical preference**: \( X \succeq Y \) if \( Q(X, Y) \geq 1/2 \)

- **Theorem**: FSD implies statistical preference

- \( \succeq \) can contain cycles:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>9</td>
<td>18</td>
</tr>
</tbody>
</table>

![Diagram](image1.png)
4. Exploiting cycle-transitivity: $T_P$

The relation $>^3_\text{P}$

$$X >^3_\text{P} Y \iff Q^\text{P}(X, Y) > \frac{\sqrt{5} - 1}{2}$$

is an asymmetric relation without cycles of length 3

The golden section $\frac{\sqrt{5}-1}{2}$: $\frac{22}{36} < \frac{\sqrt{5}-1}{2} < \frac{23}{36}$
4. A picture says more than . . .
5. Independent comparison revisited

Probabilistic relation:

\[
Q^P(X, Y) = \frac{1}{n^2} \sum_{k,l=1}^{n} \delta^P_{kl}
\]

with

\[
\delta^P_{kl} = \begin{cases} 
1, & \text{if } x_k > y_l \\
1/2, & \text{if } x_k = y_l \\
0, & \text{if } x_k < y_l 
\end{cases}
\]
5. A generalization

Probabilistic relation: \( p \in \mathbb{R}^+ \)

\[
Q^p_p(X, Y) = \frac{\sum_{k,l=1}^{n} \max(x_k - y_l, 0)^p}{\sum_{k,l=1}^{n} |x_k - y_l|^p}
\]

with \( 0/0 = 1/2 \)
5. The case of continuous RV and $p = 1$

Limit case $p = 0$: $Q_0^P = Q^P$

$Q_0^P (X, Y) = E_X[F_Y]$

since $E_X[F_Y] + E_Y[F_X] = 1$:

$$Q_0^P (X, Y) = \frac{E_X[F_Y]}{E_X[F_Y] + E_Y[F_X]}$$
5. The case of continuous RV and $p = 1$

- **Limit case** $p = 0$: $Q_0^p = Q^p$
  
  $Q_0^p(X, Y) = E_X[F_Y]$

- since $E_X[F_Y] + E_Y[F_X] = 1$:
  
  $Q_0^p(X, Y) = \frac{E_X[F_Y]}{E_X[F_Y] + E_Y[F_X]}$

- **The case** $p = 1$:
  
  - compact formula:
    
    $Q_1^p(X, Y) = \frac{E_X[G_Y]}{E_X[G_Y] + E_Y[G_X]}$

- second degree stochastic dominance?
5. Co-monotone comparison revisited

Probabilistic relation:

\[ Q^M(X, Y) = \frac{1}{n} \sum_{k=1}^{n} \delta_k^M \]

with

\[ \delta_k^M = \begin{cases} 
1 & \text{, if } x_k > y_k \\
1/2 & \text{, if } x_k = y_k \\
0 & \text{, if } x_k < y_k 
\end{cases} \]
5. Co-monotone comparison revisited

Probabilistic relation: $p \in \mathbb{R}^+$

$$Q_p^M(X, Y) = \frac{\sum_{k=1}^{n} \max(x_k - y_k, 0)^p}{\sum_{k=1}^{n} |x_k - y_k|^p}$$
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Probabilistic relation: \( p \in \mathbb{R}^+ \)

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Q_p^M(X, Y) = \frac{\sum_{k=1}^{n} \max(x_k - y_k, 0)^p}{\sum_{k=1}^{n} |x_k - y_k|^p}
\]

Limit case: \( Q_0^M = Q^M \)
5. Co-monotone comparison revisited

- Probabilistic relation: \( p \in \mathbb{R}^+ \)

\[
Q_p^M(X, Y) = \frac{\sum_{k=1}^{n} \max(x_k - y_k, 0)^p}{\sum_{k=1}^{n} |x_k - y_k|^p}
\]

- Limit case: \( Q_0^M = Q^M \)

- The case of continuous RV and \( p = 1 \):

\[
Q_1^M(X, Y) = \frac{\int \max(F_Y(x) - F_X(x), 0) \, dx}{\int |F_Y(x) - F_X(x)| \, dx}
\]
5. Graphical illustration
5. Transitivity

- **Theorem**: the probabilistic relation $Q^M_1$ is moderately stochastic transitive

- The strict order relation at $1/2$:

  $$Q^M_1(X, Y) > \frac{1}{2} \iff \mathbb{E}[X] > \mathbb{E}[Y]$$

- Any weak ($> 1/2$) or strict ($\geq 1/2$) cutting level $\alpha$ yields a strict order relation:

  - with increasing $\alpha$ the graph become more and more sparse (Hasse tree)
Thank you for your attention!

Bernard.DeBaets@UGent.be